

# ON THE COUPLING BETWEEN AN IDEAL FLUID AND IMMERSED PARTICLES

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**ABSTRACT.** In this paper we use Lagrange-Poincaré reduction to understand the coupling between a fluid and a set of Lagrangian particles that are supposed to simulate it. In particular, we reinterpret the work of Cendra et al. [CMR01] by substituting velocity interpolation from particle velocities for their principal connection. The consequence of writing evolution equations in terms of interpolation is two-fold. First, it gives estimates on the error incurred when interpolation is used to derive the evolution of the system. Second, this form of the equations of motion can inspire a family of particle and hybrid particle-spectral methods where the error analysis is “built-in”. We also discuss the influence of other parameters attached to the particles, such as shape, orientation, or higher-order deformations, and how they can help with conservation of momenta in the sense of Kelvin’s circulation theorem.

## 1. INTRODUCTION

In this paper we seek to understand, from a geometric point of view, how a set of Lagrangian particles can be used as a computational device to numerically simulate an ideal fluid. Specifically, there are certain quantities which we can associate with a finite set of particles, such as their positions, velocities, or shape change to various orders (e.g., as an evolution in  $SL(3)$ ). All these attributes may be derived by integrating the equations of motion.

However, as we point out in this paper, much insight can be gleaned by appealing to the process of Lagrange-Poincaré reduction. Specifically, given an ideal, homogeneous, inviscid, incompressible fluid on  $M$ , the configuration space may be described by the group of volume preserving diffeomorphism,  $SDiff(M)$ . If  $\odot$  is an  $N$ -tuple of distinct points in  $M$ , then we define the isotropy subgroup

$$G_{\odot} := \{\psi \in SDiff(M) \mid \psi(\odot) = \odot\}.$$

The particle relabeling symmetry of the system allows us to project the equations of motion onto the quotient space  $TSDiff(M)/G_{\odot}$ . Upon choosing an interpolation method (defined in §2.3), we obtain an isomorphism to the vector bundle  $TX \oplus \tilde{\mathfrak{g}}_{\odot}$ , where  $X$  is the configuration space of point particles and  $\tilde{\mathfrak{g}}_{\odot}$  is a vector bundle over  $X$  whose fibers are isomorphic to the Lie algebra of the symmetry group. We

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use the resulting Lagrange-Poincaré equations to describe the coupling between a fluid and its computational particles in terms of interpolation methods; we also propose a new family of particle methods.

**1.1. Previous Work.** It was shown in [Arn66] that the Euler equations of motion for an ideal, homogeneous, inviscid, incompressible fluid on an oriented Riemannian manifold  $M$  are the spatial (or Eulerian) representation of the geodesic equations on the group of volume preserving diffeomorphisms,  $\text{SDiff}(M)$ . This observation gave rise to a new perspective on fluid mechanics which lead to many developments, notably the proof of well posedness [EM70] and various extensions ranging all the way to charged fluids, magnetohydrodynamics, and even complex fluids with advected parameters (see, e.g., [Hol02], [GBR09]). All of these system are Lagrangian on the tangent bundle of groups of diffeomorphisms of a Riemannian manifold  $M$ . Additionally, all of these theories utilize the particle relabeling symmetry of the system to perform Euler-Poincaré reduction and bring the dynamics to the Lie algebra of this group [MR99, chapter 13].

We may consider reducing by subgroups of the diffeomorphism group, and there do exist frameworks for accomplishing this. This would be a special case of Lagrange-Poincaré reduction [CMR01]. In particular, we may consider reducing by isotropy groups of a set of points in  $M$ . Such an approach is already mentioned in [MD10] for the purpose of landmark matching problems; see also the references cited therein. However, to the best of our knowledge, Lagrange-Poincaré reduction has not been performed on such systems in the framework of [CMR01].

**1.2. Outline.** In §2 we establish our notation and review the notion of a generalized connection (also called an *Ehresmann connection* [MMR90]) as described in [KSM99]. We will also state a few useful formulas which result from having a generalized connection. In §2.2 we will discuss diffeomorphism groups and quotients spaces. In §2.3 we will articulate the relationship between interpolation methods and generalized connections. The equivalence of interpolation methods and principal connections is given in Appendix A. Then, in §3, we will describe the unreduced system for an inviscid fluid before reducing it by the isotropy subgroup of a finite set of particles. In §4 we will discuss reduction by higher-order isotropies which will allow us to study particles with orientation, shape, and other attributes. Finally, in §5, we will formulate a family of particle methods induced by an interpolation method and discuss some implications for the error analysis of these methods. We will find that it is possible to construct hybrid particle-spectral methods for fluids within this family. Moreover, we will show that the vortex blob algorithm fits within this family of methods and that the horizontal equations are a guide for corrections that allow for the deformation of vortex blobs. We close with §6, where we summarize how to extend these constructions to complex and other fluid models.

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## 2. PRELIMINARY MATERIAL

Before introducing our contributions, we review generalized connections and volume-preserving diffeomorphisms and prove a few important theorems.

**2.1. Generalized Connections.** In this section we introduce the notion of a generalized connection, as presented in [KSM99], and prove some useful propositions for the purpose of this paper.

**Definition 2.1.** *Let  $\pi_E : E \rightarrow M$  be a vector bundle and  $\tau_E : TE \rightarrow E$  the tangent bundle of  $E$ . Then the vertical bundle is the vector bundle  $\pi_{V(E)} : V(E) \rightarrow E$  where  $V(E) := \ker(T\pi_E)$  and  $\pi_{V(E)} := \tau_E|_{V(E)}$ .*

The vertical lift operator,  $v^\uparrow : E \oplus E \rightarrow V(E)$ , is defined by

$$v^\uparrow(v_m, w_m) := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (v_m + \epsilon w_m),$$

for all  $v_m, w_m \in E_m := \pi_E^{-1}(m)$ . This establishes an isomorphism between the vector bundles  $\text{proj}_1 : E \oplus E \rightarrow E$  and  $\pi_{V(E)} : V(E) \rightarrow E$ , where  $\text{proj}_1$  denotes the projection onto the first summand. The fiber derivative,  $\frac{\partial f}{\partial e} : E \rightarrow E^*$ , of a function  $f \in C^\infty(E)$  is defined by

$$\left\langle \frac{\partial f}{\partial e}(e), e' \right\rangle := \langle df(e), v^\uparrow(e, e') \rangle \equiv \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (f(e + \epsilon e')).$$

We shall need a notion of partial differentiation with respect to the base space  $M$ . That is, we need to understand what is meant by  $\frac{\partial f}{\partial m}$  evaluated at  $e \in E$ , as an element of  $T^*M$  over the base point  $m = \pi_E(M)$ . This is obtained by the use of a covariant derivative. For this, we define the vertical drop by  $v_\downarrow : V(E) \rightarrow E := \text{proj}_2 \circ (v^\uparrow)^{-1}$ , where  $\text{proj}_2 : E \oplus E \rightarrow E$  is the projection onto the second component.

**Definition 2.2.** Let  $\pi_E : E \rightarrow M$  be a vector bundle. A horizontal bundle is a subbundle,  $H(E) \subset TE$ , such that  $TE = H(E) \oplus V(E)$ . This defines projectors  $\text{hor} : TE \rightarrow H(E)$  and  $\text{ver} : TE \rightarrow V(E)$ . The vertical projector,  $\text{ver}$ , is called a generalized connection. If  $C(I; E)$  denotes the set of smooth curves in  $E$  on an open interval  $I \subset \mathbb{R}$ , then the covariant derivative induced by  $H(E)$  (or  $\text{ver}$ ) is the map  $\frac{D}{Dt} : C(I; E) \rightarrow E$  given by

$$\frac{De}{Dt} = v_\downarrow \left( \text{ver} \left( \frac{de}{dt} \right) \right)$$

The choice of a generalized connection on a vector bundle  $\pi_E : E \rightarrow M$  induces the *horizontal lift* operator  $h^\uparrow : E \oplus TM \rightarrow H(E)$  by the condition that  $h^\uparrow(e, \dot{m})$  is the unique horizontal vector in the fiber  $T_e E$  such that  $T\pi_E(h^\uparrow(e, \dot{m})) = \dot{m}$ , where  $e \in E$  and  $\dot{m} \in T_m M$ . This implies that

$$h^\uparrow := \left[ (\tau_E \oplus T\pi_E)|_{H(E)} \right]^{-1}.$$

Thus, the choice of a generalized connection introduces the partial derivative of the function  $f \in C^\infty(E)$  with respect to  $m$  as a vector bundle map  $\frac{\partial f}{\partial m} : E \rightarrow T^*M$  over the base  $M$  defined by

$$\left\langle \frac{\partial f}{\partial m}(e), \delta m \right\rangle := \langle df(e), h^\uparrow(e, \delta m) \rangle,$$

for all  $e \in E$  and  $\delta m \in T_m M$ .

By construction, this means that the total exterior derivative  $df$  acting on the velocity of a curve  $e(t) \in E$  over  $m(t) \in M$  be written as the sum

$$(1) \quad \left\langle df(e), \frac{de}{dt} \right\rangle = \left\langle \frac{\partial f}{\partial m}(e), \frac{dm}{dt} \right\rangle + \left\langle \frac{\partial f}{\partial e}(e), \frac{De}{Dt} \right\rangle.$$

If  $M$  is a Riemannian manifold we may use the Levi-Civita connection, defined on the vector bundle  $TM$ . The torsion free property of the Levi-Civita connection is equivalent to the statement that  $\Gamma_{ij}^k = \Gamma_{ji}^k$ , where  $\Gamma_{ij}^k$  are Christoffel symbols of the connection in an arbitrary local coordinate system. The covariant derivative with respect to the Levi-Civita connection is given (locally) by  $\frac{Dv^k}{Dt} = \frac{dv^k}{dt} + \Gamma_{ij}^k v^i \frac{dm^j}{dt}$ , where  $m^j$  denote local coordinates of the point  $m \in M$ . Therefore, the horizontal lift induced by the Levi-Civita connection is given locally by

$$(2) \quad h^\uparrow((m^i, v^j), (m^i, \delta m^j)) = ((m^i, v^j), (\delta m^l, -\Gamma_{ij}^k v^i \delta m^j)).$$

This allows us to easily prove the following proposition.

**Proposition 2.1.** Let  $\alpha \in \Omega^1(M)$ . A torsion free connection on  $TM$  (such as a Levi-Civita connection) induces the identity

$$(3) \quad d\alpha(m)(v, w) = \left\langle \frac{\partial \alpha}{\partial m}(w), v \right\rangle - \left\langle \frac{\partial \alpha}{\partial m}(v), w \right\rangle,$$

where  $v, w \in T_m M$ .

*Proof.* This may be verified in a local coordinate chart, where  $\alpha(m) = \alpha_i(m) dm^i$ . Viewing  $\alpha$  as a function on  $TM$  we find that equation (2) implies

$$\left\langle \frac{\partial \alpha}{\partial m}(m^i, w^j), v^k \frac{\partial}{\partial m^k} \right\rangle = \frac{\partial \alpha_i}{\partial m^j}(m) w^i v^j - \alpha_k(m) \Gamma_{ij}^k(m) w^i v^j.$$

Since  $d\alpha(m)(v, w) = \left( \frac{\partial \alpha_i}{\partial m^j} - \frac{\partial \alpha_j}{\partial m^i} \right) v^j w^i$ , we can see that (3) follows from the torsion free property,  $\Gamma_{ij}^k = \Gamma_{ji}^k$ .  $\square$

This proof is extended to infinite dimensional manifolds by noting the irrelevance of the local coordinate description of  $\Gamma_{ij}^k$ . It can also be extended to one-forms on  $M$  with values in a vector bundle  $E \rightarrow M$  by defining the exterior derivative on a tensor product by

$$d(e \otimes \alpha) := e \otimes d\alpha$$

where  $e$  is a section of the vector bundle  $E \rightarrow M$  and  $\alpha \in \Omega^1(M)$  is an ordinary one-form on  $M$ .

**2.2. Diffeomorphism Groups.** Let  $M$  be a finite dimensional connected Riemannian manifold with metric  $\langle \cdot, \cdot \rangle_M : TM \oplus TM \rightarrow \mathbb{R}$ . The set of volume preserving diffeomorphisms,  $\text{SDiff}(M)$ , of  $M$  is an infinite dimensional Fréchet Lie group. A tangent vector  $v_\varphi \in T_\varphi \text{SDiff}(M)$  over a group element  $\varphi \in \text{SDiff}(M)$  is a map  $v_\varphi : M \rightarrow TM$  such that  $v_\varphi(m) \in T_{\varphi(m)} M$ . The Lie algebra  $\mathfrak{X}_{\text{div}}(M)$  of  $\text{SDiff}(M)$  is the set of divergence free vector fields on  $M$ . Therefore,  $T_\varphi \text{SDiff}(M) = \{u \circ \varphi \mid u \in \mathfrak{X}_{\text{div}}(M)\}$ .

Let  $\odot = (\odot_1, \dots, \odot_N) \in M^N$  be an  $N$ -tuple of distinct points in  $M$ . We define the isotropy group

$$G_\odot := \{\psi \in \text{SDiff}(M) \mid \psi(\odot) = \odot\},$$

where  $\text{SDiff}(M)$  acts on  $\odot \in M^N$  by the diagonal action. It is elementary to see that the quotient of  $\text{SDiff}(M)$  by the right action of  $G_\odot$  is the set

$$X := \{(m_1, \dots, m_N) \in M^N \mid m_i \neq m_j\}.$$

We shall define in §3 a Lagrangian system on  $T \text{SDiff}(M)$ , invariant under right translations by elements of  $G_\odot$  and implement Lagrange-Poincaré reduction along the lines of [CMR01]. This task is less trivial than it may first sound because  $(T \text{SDiff}(M))/G_\odot \neq T(\text{SDiff}(M)/G_\odot)$ . We denote elements of  $T \text{SDiff}(M)/G_\odot$  by  $[v_\varphi]$  for each  $v_\varphi \in T \text{SDiff}(M)$ . We will find that  $(T \text{SDiff}(M))/G_\odot$  is a vector bundle over  $X$ . Thus, Lagrange-Poincaré reduction will provide dynamics which describe how the motion of a finite set of particles couples to the fluid.

**2.3. Interpolation Methods as Generalized Connections.** Define the vector bundle

$$\tilde{\mathfrak{g}}_\odot := \{(x, \xi) \mid x \in X, \xi \in \mathfrak{g}_x\}$$

where  $\mathfrak{g}_x$  is the Lie algebra of the isotropy group  $G_x$  for each  $x \in X$ , i.e., the set of divergence free vector fields which vanish at  $x \in X \subset M^N$ . As a side note, the bundle  $\tilde{\mathfrak{g}}_\odot$  is identical to the adjoint bundle  $\frac{\text{SDiff}(M) \times \mathfrak{g}_\odot}{G_\odot}$  when equipped with the fiberwise Lie bracket  $[(x, \xi_x), (x, \eta_x)] := (x, [\xi_x, \eta_x]_{\text{Jacobi-Lie}})$  and the projection  $\tilde{\pi}(x, \xi_x) = x$ .

We will ultimately identify the quotient  $(T \text{SDiff}(M))/G_\odot$  with  $TX \oplus \tilde{\mathfrak{g}}_\odot$ .

**Definition 2.3.** A  $\mathfrak{X}_{\text{div}}(M)$ -valued one-form  $\mathcal{I} : TX \rightarrow \mathfrak{X}_{\text{div}}(M)$  on  $X$  such that  $\mathcal{I}(\dot{x})(x) = \dot{x}$  for  $\dot{x} \in TX$  and  $x = \tau_X(\dot{x})$  is called an interpolation method.

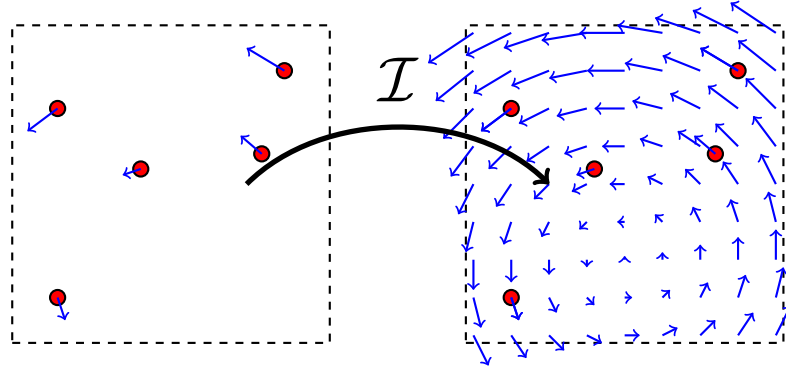


FIGURE 1. Schematic representation of an Interpolation Method

**Example 2.1.** Initially, one would desire that  $\int_M \langle \mathcal{I}(\dot{x})(m), \xi_x(m) \rangle d\text{vol}(m) = 0$  for all  $(x, \xi_x) \in \tilde{\mathfrak{g}}_\odot$ . Such an interpolation method would correspond to choosing the “mechanical connection” (see [MMR90] section 2.4). However, this interpolation method does not exist. Indeed, if such an interpolation method did exist, then  $\mathcal{I}(\dot{x})$  for some non-zero velocity  $\dot{x} \in TX$  would need to vanish everywhere, yet be non-zero at the locations of some of the particles. This would not define something which is smooth. However, if we choose an inner product which makes  $\mathfrak{X}_{\text{div}}(M)$  into a Hilbert space we may be able to do something more systematic. For example if  $M = \mathbb{R}^d$ , and  $\mathbb{I} : \mathfrak{X}(\mathbb{R}^d) \rightarrow \mathfrak{X}(\mathbb{R}^d)$  is a positive definite SE(3)-invariant operator, we may define the inner product

$$\langle u, v \rangle_{\mathbb{I}} = \int_{\mathbb{R}^d} u(m) \cdot [\mathbb{I}v](m) d\text{vol}(m)$$

with associated reconstruction mapping

$$\mathcal{I}(\dot{x})(m) = \sum_i G(|m - x_i|) \dot{x}^i,$$

where  $G(|x|)$  is Green's function of  $\mathbb{I}$  on the space  $\mathfrak{X}(\mathbb{R}^d)$ . For example, if  $\mathbb{I} = 1 - \alpha \cdot \Delta$  for some  $\alpha > 0$ , then  $G(|x|) = \exp(-|x|/\alpha)$ . This is not directly applicable in the context of ideal incompressible fluids since this interpolation method does not produce divergence free vector fields and the operator  $1 - \alpha \cdot \Delta$  is not naturally identified with ideal fluids. However, this construction is natural in the case of the EPDiff equation with respect to the Lagrangian induced by the  $H^1$  norm (see [HSS09, part II] and reference therein).

Returning to the general case and using coordinates  $(x, \xi_x, \dot{x}, \dot{\xi}_x)$  for  $T\tilde{\mathfrak{g}}_\odot$ , we can see that the choice of an interpolation method  $\mathcal{I}$  induces a horizontal space for the vector bundle  $\tilde{\mathfrak{g}}_\odot$  given by elements of the form

$$(x, \xi_x, \dot{x}, [\xi_x, \mathcal{I}(\dot{x})]) \in T\tilde{\mathfrak{g}}_\odot.$$

In particular, we obtain the horizontal and vertical projections

$$\begin{aligned} \text{hor}(x, \xi_x, \dot{x}, \dot{\xi}_x) &= (x, \xi_x, \dot{x}, [\xi_x, \mathcal{I}(\dot{x})]) \\ \text{ver}(x, \xi_x, \dot{x}, \dot{\xi}_x) &= (x, \xi_x, 0, \dot{\xi}_x - [\xi_x, \mathcal{I}(\dot{x})]). \end{aligned}$$

**Proposition 2.2.** *The covariant derivative induced by the horizontal space induced by an interpolation method,  $\mathcal{I}$ , is given by*

$$\frac{D(x, \xi_x)}{Dt} = \left( x, \frac{d\xi_x}{dt} - \left[ \xi_x, \mathcal{I} \left( \frac{dx}{dt} \right) \right] \right).$$

*Proof.* Since  $v = \frac{d}{dt}(x, \xi_x)$ ,  $\text{ver}(v) = v - \text{hor}(v)$ , and  $\text{hor}(v) = (x, \xi_x, \dot{x}, [\xi_x, \mathcal{I}(\dot{x})])$ , applying Definition 2.2 for the covariant derivative induced by a horizontal bundle, we conclude

$$\frac{D(x, \xi_x)}{Dt} = v_\downarrow \left( \text{ver} \left( \frac{d}{dt}(x, \xi_x) \right) \right) = \left( x, \frac{d\xi_x}{dt} - \left[ \xi_x, \mathcal{I} \left( \frac{dx}{dt} \right) \right] \right),$$

as stated.  $\square$

The formula in Proposition 2.2 immediately yields the expression of the covariant derivative on the dual bundle,  $\tilde{\mathfrak{g}}_\odot^*$ , namely,

$$\frac{D}{Dt}(x, \alpha_x) = \left( x, \frac{d\alpha_x}{dt} - \text{ad}_{\mathcal{I}(\dot{x})}^*(\alpha_x) \right).$$

This will be useful later when we take covariant derivatives of momenta.

**Proposition 2.3.** *Given an interpolation method,  $\mathcal{I} : TX \rightarrow \mathfrak{X}_{\text{div}}(M)$ , the map  $\Psi_{\mathcal{I}} : T\text{SDiff}(M)/G_\odot \rightarrow TX \oplus \tilde{\mathfrak{g}}_\odot$  given by*

$$\Psi_{\mathcal{I}}([v_\varphi]) = (v_\varphi(\odot), v_\varphi \circ \varphi^{-1} - \mathcal{I}(v_\varphi(\odot)))$$

*is an isomorphism of vector bundles.*

*Proof.* We first must show that  $\Psi_{\mathcal{I}}$  is well defined on the quotient space  $T \text{SDiff}(M)/G_{\odot}$ . Let  $\psi \in G_{\odot}$ ; note that

$$\begin{aligned}\Psi_{\mathcal{I}}([v_{\varphi} \circ \psi]) &= (v_{\varphi}(\psi(\odot)), (v_{\varphi} \circ \psi) \circ (\varphi \circ \psi)^{-1} - \mathcal{I}(v_{\varphi}(\psi(\odot)))) \\ &= (v_{\varphi}(\odot), v_{\varphi} \circ \varphi^{-1} - \mathcal{I}(v_{\varphi}(\odot))) \\ &= \Psi_{\mathcal{I}}([v_{\varphi}]).\end{aligned}$$

Additionally, it is easy to check that  $\Psi_{\mathcal{I}}$  has the inverse  $\Psi_{\mathcal{I}}^{-1}(x, \dot{x}, \xi_x) = [u \circ \varphi]$  where  $u = \xi_x + \mathcal{I}(\dot{x})$ .  $\square$

### 3. LAGRANGIAN REDUCTION AND THE EQUATIONS OF MOTION

The kinetic energy of a fluid flowing on  $M$ , denoted  $L : T \text{SDiff}(M) \rightarrow \mathbb{R}$ , is given by

$$(4) \quad L(\varphi, \dot{\varphi}) = \frac{\rho}{2} \int_M \|\dot{\varphi}(m)\|^2 d\text{vol}(m),$$

where  $\rho$  denotes the density of the fluid, assumed to be constant. It was shown in [Arn66] that  $L$  is  $\text{SDiff}(M)$ -invariant and that the resulting Euler-Poincaré equations are precisely Euler's equations for an ideal, inviscid, homogeneous, incompressible fluid

$$\begin{aligned}\frac{\partial u}{\partial t} + \nabla_u u &= -\frac{\nabla p}{\rho} \\ \text{div } u &= 0\end{aligned}$$

for any oriented boundaryless Riemannian manifold  $M$ . In this section we shall take  $M = \mathbb{R}^d$  and will do a reduction by  $G_{\odot} \subset \text{SDiff}(M)$ . Of course, the resulting equations of motion yield the same dynamics. However, the use of the interpolation method heavily influences how one writes the equations of motion. To see this, we must first study how variations of curves in  $\text{SDiff}(M)$  lead to variations of curves in  $TX \oplus \tilde{\mathfrak{g}}_{\odot}$  under the map  $\Psi_{\mathcal{I}}$  induced by an interpolation method,  $\mathcal{I}$ .

**3.1. Covariant Variations.** Let  $\varphi_t$  be a curve in  $\text{SDiff}(M)$ . Then a deformation is a surface embedding,  $\varphi_{\lambda,t}$ , such that  $\varphi_{0,t} = \varphi_t$ . We desire to measure how much the variation  $\delta\varphi_t := \left. \frac{\partial \varphi_{\lambda,t}}{\partial \lambda} \right|_{\lambda=0}$  changes the quantity  $(x, \xi_x) := (x, \dot{\varphi} \circ \varphi^{-1} - \mathcal{I}(\dot{x})) \in \tilde{\mathfrak{g}}_{\odot}$  where  $x(t) = \varphi_t(\odot)$  and  $\dot{\varphi}_t = \frac{d\varphi_t}{dt}$ . To do this, we will invoke the covariant derivative induced by  $\mathcal{I}$ .

Additionally, through the Riemannian metric,  $\langle \cdot, \cdot \rangle_X$ , on  $X$  given by

$$\langle v_x, w_x \rangle_X := \int_M \langle \mathcal{I}(v_x)(m), \mathcal{I}(w_x)(m) \rangle_M d\text{vol}(m),$$

we may use the resulting Levi-Civita connection to get a covariant derivative on the Whitney sum  $TX \oplus \tilde{\mathfrak{g}}_{\odot}$ . This direct sum of covariant derivatives will also be denoted  $\frac{D}{Dt}$ .



Using this covariant derivative, we may define the *covariant variation* of a curve  $(x, \xi_x)(t) \in \tilde{\mathfrak{g}}_\odot$  with respect to a deformation  $(x, \xi_x)_{\lambda,t}$  by  $\delta^{\mathcal{I}}(x, \xi_x) := \left. \frac{D(x, \xi_x)}{D\lambda} \right|_{\lambda=0}$ . However, we will primarily be concerned with variations induced by variations of curves in  $\text{SDiff}(M)$ . The following propositions describe the form of such variations.

**Proposition 3.1.** *Let  $\varphi_t$  be a curve in  $\text{SDiff}(M)$ . Set  $\dot{\varphi}_t = \frac{d}{dt}(\varphi_t) \in T_{\varphi_t} \text{SDiff}(M)$ ,  $x(t) = \varphi_t(\odot)$ ,  $\dot{x}(t) = \dot{\varphi}_t(\odot) \equiv \frac{dx}{dt}$ . Finally, set*

$$\xi_x = \dot{\varphi} \circ \varphi^{-1} - \mathcal{I}(\dot{x}),$$

*so that we have a curve  $(x, \xi_x)(t) \in \tilde{\mathfrak{g}}_\odot$ . Then, provided a vertical deformation of  $\varphi_t$  given by  $\varphi_{t,\lambda} = \varphi_t \circ \psi_{t,\lambda}$  for a deformation  $\psi_{t,\lambda} \in G_\odot$  where  $\psi_{t,0}$  is the identity, the covariant variation of  $(x, \xi_x)(t)$  is given by*

$$\delta^{\mathcal{I}}(x, \xi_x) = \frac{D(x, \eta_x)}{Dt} + [(x, \xi_x), (x, \eta_x)]$$

where  $\eta_x = \varphi_* \eta$  and  $\eta := \left. \frac{\partial \psi_{t,\lambda}}{\partial \lambda} \right|_{\lambda=0} \in \mathfrak{g}_\odot$ .

*Proof.* We decompose  $\delta \xi_x := \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} (\xi_x)$  into three parts:

$$\delta \xi_x = \underbrace{\frac{\partial \dot{\varphi}}{\partial \lambda} \circ \varphi^{-1}}_{T_1} + \underbrace{\dot{\varphi} \circ \frac{\partial \varphi^{-1}}{\partial \lambda}}_{T_2} + \underbrace{\frac{\partial \mathcal{I}(\dot{x})}{\partial \lambda}}_{T_3}.$$

As  $\mathcal{I}(\dot{x})$  does not depend on  $\lambda$  we may set  $T_3$  equal to 0. We can rewrite  $T_1$  as

$$\begin{aligned} T_1 &= \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} (\dot{\varphi}) \circ \varphi^{-1} = \frac{\partial^2 \varphi}{\partial \lambda \partial t} \circ \varphi^{-1} \\ &= \frac{\partial}{\partial t} (\delta \varphi) \circ \varphi^{-1} \\ &= \frac{\partial}{\partial t} (T\varphi \circ \eta) \circ \varphi^{-1} \\ &= T\dot{\varphi} \circ \eta \circ \varphi^{-1} + T\varphi \circ \dot{\eta} \circ \varphi^{-1} \\ &= T\dot{\varphi} \circ \eta \circ \varphi^{-1} + \varphi_* \dot{\eta}, \end{aligned}$$

while  $T_2$  may be written as

$$\begin{aligned} T_2 &= T\dot{\varphi} \circ \frac{\partial \varphi^{-1}}{\partial \lambda} = -T\dot{\varphi} \circ \varphi^{-1} \circ \delta \varphi \circ \varphi^{-1} \\ &= -T\dot{\varphi} \circ \varphi^{-1} \circ \varphi \circ \eta \circ \varphi \\ &= -T\dot{\varphi} \circ \eta \circ \varphi^{-1} \end{aligned}$$

Thus, we find  $\delta \xi_x = \varphi_* \dot{\eta}$ . Since  $\delta x = 0$ , we see that

$$\delta^{\mathcal{I}}(x, \xi_x) = (x, \delta \xi_x + [\mathcal{I}(\delta x), \xi_x]) = (x, \varphi_* \dot{\eta}).$$

Additionally, note that

$$\frac{D(x, \eta_x)}{Dt} = \left( x, \frac{d\eta_x}{dt} + [\mathcal{I}(\dot{x}), \eta_x] \right).$$

We calculate

$$\begin{aligned} \frac{d\eta_x}{dt} &= \frac{d}{dt} \varphi_* \eta = -[\dot{\varphi} \circ \varphi^{-1}, \varphi_* \eta] + \varphi_* \dot{\eta} \\ &= -[\xi + \mathcal{I}(\dot{x}), \eta_x] + \varphi_* \dot{\eta}. \end{aligned}$$

Therefore  $\varphi_* \dot{\eta} = \frac{d\eta_x}{dt} + [\xi + \mathcal{I}(\dot{x}), \eta_x]$  so that

$$\delta^{\mathcal{I}}(x, \xi_x) = \left( x, \frac{d\eta_x}{dt} + [\xi_x + \mathcal{I}(\dot{x}), \eta_x] \right) = \frac{D(x, \eta_x)}{Dt} + [(x, \xi_x), (x, \eta_x)]$$

as required.  $\square$

While vertical variations are now clear, we must also consider how  $(x, \xi_x) = (x, \dot{\varphi} \circ \varphi^{-1} - \mathcal{I}(\dot{x}))$  varies in response to variations of  $x \in X$ . Given a curve  $\varphi_t \in \text{SDiff}(M)$  we may take a deformation of the curve  $x_t = \varphi_t(\odot) \in X$  given by  $x_{t,\lambda}$ . If we consider the sequence of vector fields  $\mathcal{I}\left(\frac{\partial x_{t,\lambda}}{\partial \lambda}\right)$  parameterized by  $\lambda$  we may integrate them over  $\lambda$  to get a diffeomorphism. This defines the *horizontal deformation* of  $\varphi_t$  by

$$(5) \quad \varphi_{t,\lambda} = \int_0^\lambda \mathcal{I}\left(\frac{\partial x_{t,\lambda'}}{\partial \lambda'}\right) d\lambda' \circ \varphi_t.$$

**Proposition 3.2.** *Let  $\varphi_t$  be a curve in  $\text{SDiff}(M)$  and let  $x(t) = \varphi_t(\odot)$ . Let  $x_{t,\lambda}$  be a deformation of  $x(t)$  and let  $\varphi_{t,\lambda}$  be the resulting horizontal deformation given by equation (5). Then the covariant variation of  $(x, \xi_x)$  where  $\xi_x := \dot{\varphi} \circ \varphi^{-1} - \mathcal{I}(\dot{x})$  is given by*

$$\delta^{\mathcal{I}}(x, \xi_x) = \tilde{B}(\dot{x}, \delta x),$$

where

$$(6) \quad \tilde{B}(\dot{x}, \delta x) = (x, d\mathcal{I}(\dot{x}, \delta x) - [\mathcal{I}(\dot{x}), \mathcal{I}(\delta x)]).$$

*Proof.* Upon invoking equation (1) and the fact that the fiber derivative,  $\frac{\partial \mathcal{I}}{\partial \dot{x}}$ , is identical to  $\mathcal{I}$  (because  $\mathcal{I}$  is linear on fibers of  $TX$ ) we find that

$$\delta \xi_x = \left. \frac{\partial \dot{\varphi}}{\partial \lambda} \right|_{\lambda=0} \circ \varphi^{-1} - \dot{\varphi} \circ \varphi^{-1} \circ \delta \varphi \circ \varphi^{-1} - \left\langle \frac{\partial \mathcal{I}}{\partial \dot{x}}(\dot{x}), \delta x \right\rangle - \mathcal{I}\left(\frac{D}{Dt}(\delta x)\right).$$

Using equality of mixed partials we find that

$$\begin{aligned} \left. \frac{\partial \dot{\varphi}}{\partial \lambda} \right|_{\lambda=0} \circ \varphi^{-1} &= \frac{d}{dt}(\delta \varphi) \circ \varphi^{-1} \\ &= \frac{d}{dt}(\mathcal{I}(\delta x) \circ \varphi) \circ \varphi^{-1} \\ &= \left\langle \frac{\partial \mathcal{I}}{\partial x}(\delta x), \dot{x} \right\rangle + \mathcal{I} \left( \frac{D}{Dt}(\delta x) \right) - \mathcal{I}(\delta x) \circ \dot{\varphi} \circ \varphi^{-1} \end{aligned}$$

Upon substitution into the last line of the previous calculation we find

$$\begin{aligned} \delta \xi_x &= \left( \left\langle \frac{\partial \mathcal{I}}{\partial x}(\delta x), \dot{x} \right\rangle - \left\langle \frac{\partial \mathcal{I}}{\partial x}(\dot{x}), \delta x \right\rangle \right) + \mathcal{I}(\delta x) \circ (\dot{\varphi} \circ \varphi^{-1}) + (\dot{\varphi} \circ \varphi^{-1}) \circ \mathcal{I}(\delta x) \\ &\stackrel{(3)}{=} d\mathcal{I}(\dot{x}, \delta x) + [\mathcal{I}(\delta x), \dot{\varphi} \circ \varphi^{-1}] \\ &= d\mathcal{I}(\dot{x}, \delta x) + [\mathcal{I}(\delta x), \xi_x + \mathcal{I}(\dot{x})]. \end{aligned}$$

Therefore, the covariant variation is

$$\delta^{\mathcal{I}}(x, \xi_x) = (x, \delta \xi_x - [\mathcal{I}(\delta x), \xi_x]) = d\mathcal{I}(\dot{x}, \delta x) - [\mathcal{I}(\dot{x}), \mathcal{I}(\delta x)] := \tilde{B}(\dot{x}, \delta x)$$

as stated.  $\square$

We call  $\tilde{B}$  the *reduced curvature tensor*. It measures the non-integrability of the horizontal distribution induced by the interpolation method. This becomes even more clear in the following proposition.

**Proposition 3.3.** *The reduced curvature tensor (6) may equivalently be written as*

$$\tilde{B}(\dot{x}, \delta x) = \mathcal{I}([\mathcal{I}(\dot{x}), \mathcal{I}(\delta x)](x)) - [\mathcal{I}(\dot{x}), \mathcal{I}(\delta x)].$$

The proof is built on the standard method used for the curvature tensor associated to a principal connection. In order to preserve the flow of the presentation we have included the proof in Appendix A.

As a result of Propositions 3.1 and 3.2, we see that the most general covariant variations of  $(x, \xi_x) \in \tilde{\mathfrak{g}}_{\odot}$  that are needed in the variational principle, are of the form

$$(7) \quad \delta^{\mathcal{I}}(x, \xi_x) = \frac{D(x, \eta_x)}{Dt} + [(x, \xi_x), (x, \eta_x)] + \tilde{B}(\dot{x}, \delta x)$$

for some curve  $(x, \eta_x) \in \tilde{\mathfrak{g}}_{\odot}$  and a variation  $\delta x(t)$  of the curve  $x(t)$ .

**3.2. Lagrange-Poincaré Reduction.** In this section, we state the Lagrange-Poincaré reduction theorem (see [CMR01]) in terms of interpolation methods rather than principal connections. The resulting equations of motion induce the

same dynamics, of course. Note that the reduced Lagrangian,  $[L]$ , written on  $TX \oplus \tilde{\mathfrak{g}}$  via the isomorphism  $\Psi_{\mathcal{I}}$  is given by

$$(8) \quad [L](x, \dot{x}, \xi_x) = \frac{\rho}{2} \int_M \|\xi_x + \mathcal{I}(\dot{x})\|^2 d\text{vol}(m).$$

**Theorem 3.1.** *Let  $L : T\text{SDiff}(M) \rightarrow \mathbb{R}$  be a  $G_{\odot}$  invariant Lagrangian such as the kinetic energy Lagrangian given in equation (4). Let  $\mathcal{I} : TX \rightarrow \mathfrak{X}_{\text{div}}(M)$  be an interpolation method, and let  $[L] : TX \oplus \tilde{\mathfrak{g}}_{\odot} \rightarrow \mathbb{R}$  be the reduced Lagrangian. Let  $\varphi_t \in \text{SDiff}(M)$  be a curve and set  $x(t) = \varphi_t(\odot)$ ,  $\xi_x = \dot{\varphi}_x \circ \varphi^{-1} - \mathcal{I}(\dot{x})$ . Then the following are equivalent.*

- (1) *The curve  $\varphi_t$  is critical for the action*

$$S = \int_0^t L(\varphi_t, \dot{\varphi}_t) dt$$

*with respect to variations  $\delta\varphi_t$  with fixed end points.*

- (2) *The curve  $\varphi_t$  satisfies the Euler-Lagrange equations*

$$\frac{D}{Dt} \left( \frac{\partial L}{\partial \dot{\varphi}} \right) - \frac{\partial L}{\partial \varphi} = 0.$$

*with respect to an arbitrary covariant derivative and connection on  $T\text{SDiff}(M)$ .*

- (3) *The curve  $(x, \dot{x}, \xi_x)(t)$  is critical for the reduced action*

$$[S] = \int_0^t [L](x, \dot{x}, \xi_x) dt$$

*with respect to arbitrary variations  $\delta x(t)$ , with fixed end points, and covariant variations of  $(x, \xi_x)$  of the form*

$$\delta \mathcal{I}(x, \xi_x) = \frac{D(x, \eta_x)}{Dt} - [(x, \eta_x), (x, \xi_x)] + \tilde{B}(\dot{x}, \delta x)$$

*for some curve  $(x, \eta_x) \in \tilde{\mathfrak{g}}_{\odot}$ .*

- (4) *The curve  $(x, \dot{x}, \xi_x)$  satisfies the Lagrange-Poincaré equations*

$$\begin{aligned} \frac{D}{Dt} \left( \frac{\partial [L]}{\partial \dot{x}} \right) - \frac{\partial [L]}{\partial x} &= i_{\dot{x}} \tilde{B}_{\mu} \quad \text{on } T_{x(t)}X \\ \frac{D}{Dt} \left( \frac{\partial [L]}{\partial \xi_x} \right) &= -\text{ad}_{\xi_x}^* \left( \frac{\partial [L]}{\partial \xi_x} \right) \quad \text{on } \tilde{\mathfrak{g}}_{\odot}(x(t)) \end{aligned}$$

$$\text{where } \tilde{B}_{\mu}(v_x, w_x) = \left\langle \frac{\partial [L]}{\partial \xi_x}, \tilde{B}(v_x, w_x) \right\rangle.$$

*Proof.* The equivalence of (1) and (2) is an intrinsic formulation of the standard derivation of the Euler-Lagrange equations such as done in [AM00]. We have chosen to write the intrinsic formulation more out of necessity than interest, as we are working on a space with non-trivial coordinate charts. In this case, the

“equivalence of mixed partials” comes from our definition of the covariant derivative induced by a generalized connection. Specifically, let  $\varphi_{t,\epsilon}$  be an embedding of a surface into  $\text{SDiff}(M)$  (i.e., a deformation of a curve). Then we observe that

$$(9) \quad \frac{D\dot{\varphi}_t}{D\epsilon} = v_\downarrow \left( \text{ver} \left( \frac{\partial^2 \varphi_{t,\epsilon}}{\partial t \partial \epsilon} \Big|_{\epsilon=0} \right) \right) = \frac{D\delta\varphi_t}{Dt}$$

where  $\dot{\varphi}_t = \frac{\partial \varphi_{t,0}}{\partial t}$  and  $\delta\varphi_t = \frac{\partial \varphi_{t,\epsilon}}{\partial \epsilon} \Big|_{\epsilon=0}$ . Using this, one is able to prove the equivalence of (1) and (2) in the standard way using integration by parts.

We now prove the equivalence of (1) and (3). If the time integral of  $L$  is extremized along  $(\varphi, \dot{\varphi})_t$  then  $[L]$  must be extremized along  $\Psi^{\mathcal{I}}([\varphi_t, \dot{\varphi}_t])$ . However, by proposition 3.2 and 3.1, we know that it must be extremized with respect to variations of the form given in equation (7).

Finally, we prove the equivalence of (3) and (4). Assume  $[S]$  is extremized with respect to the variations given in equation (7). Then we find, with equation (1), that

$$\begin{aligned} \delta[S] &= \frac{\partial}{\partial \lambda} \Big|_{\lambda=0} \int_0^t [L](x_\lambda, \dot{x}_\lambda, \dot{\xi}_{x,\lambda}) dt \\ &= \int_0^t \langle d[L], \delta(x, \dot{x}, \xi_x) \rangle dt \\ &= \int_0^t \left\langle \frac{\partial[L]}{\partial \xi_x}, \delta^{\mathcal{I}} \xi_x \right\rangle + \left\langle \frac{\partial[L]}{\partial \dot{x}}, \frac{D\dot{x}}{D\epsilon} \right\rangle + \left\langle \frac{\partial[L]}{\partial x}, \delta x \right\rangle dt. \end{aligned}$$

Using the definition of the covariant derivative and equality of mixed partials we get

$$\frac{D\dot{x}}{D\epsilon} = v_\downarrow \left( \text{ver} \left( \frac{\partial^2 x}{\partial t \partial \epsilon} \right) \right) = \frac{D\delta x}{Dt}$$

so that the last calculation becomes

$$\begin{aligned} \delta[S] &= \int_0^t \left\langle \frac{\partial[L]}{\partial \xi_x}, \frac{D(x, \eta_x)}{Dt} + [(x, \eta_x), (x, \xi_x)] + \tilde{B}(\dot{x}, \delta x) \right\rangle dt \\ &\quad \int_0^t \left\langle \frac{\partial[L]}{\partial \dot{x}}, \frac{D\delta x}{Dt} \right\rangle + \left\langle \frac{\partial[L]}{\partial x}, \delta x \right\rangle dt \\ &= \int_0^t - \left\langle \frac{D}{Dt} \left( \frac{\partial[L]}{\partial \xi_x} \right) - \text{ad}_{\xi_x}^* \left( \frac{\partial[L]}{\partial \xi_x} \right), (x, \eta_x) \right\rangle dt \\ &\quad + \int_0^t \left\langle \frac{D}{Dt} \left( \frac{\partial[L]}{\partial \dot{x}} \right) - \frac{\partial[L]}{\partial x} - i_{\dot{x}} \tilde{B}_\mu, \delta x \right\rangle dt. \end{aligned}$$

The theorem follows from the arbitrariness of  $(x, \eta_x)$  and  $\delta x$  on the interior of the interval. Additionally, the above sequence of calculations is reversible.  $\square$

At this point, one may be inspired to come up with particle methods by trying to better understand the horizontal equation (the equation for the dynamics on  $X$ ).

Alternatively, others may be very skeptical of this idea since the vertical equation is nearly everything. In fact if we unpack the terms of the vertical equation, we find that  $\frac{\partial[L]}{\partial \xi_x} = \langle \xi_x + \mathcal{I}(\dot{x}), \cdot \rangle_{L^2} = u^b$ , so that the left hand side is

$$\frac{Du^b}{Dt} = \frac{\partial u^b}{\partial t} + \text{ad}_{\mathcal{I}(\dot{x})}^*(u^b),$$

while the right hand side of the vertical equation is  $-\text{ad}_{(x, \xi_x)}^*(u^b)$ . Bringing both terms to one side we find  $\frac{\partial u^b}{\partial t} + \text{ad}_u^* u^b = 0$ , which is the inviscid fluid equation [AK92], except for the fact that the vertical equations (strictly speaking) only address the domain  $M \setminus x$ . In essence, the horizontal equations only state that the particles move in such a way that  $u$  can be extended smoothly by “filling the hole”. Despite this sobering observation, we know that computational scientists simulate fluids and successfully use interpolation methods all the time. After studying what happens when we reduce by a class of subgroups of  $G_\odot$  in §4, we will try to understand how the horizontal equation can potentially inspire particle methods in §5.

#### 4. HIGHER ORDER ISOTROPY GROUPS

In the previous section we reduced our system by the Lie group  $G_\odot$ . In this section, we consider the Lie group

$$G_\odot^{(k)} := \{\psi \in G_\odot \mid T_\odot^k \psi \text{ is the identity on } T_\odot^k M\}.$$

In local coordinates, elements of  $G_\odot^{(k)}$  are diffeomorphisms such that the Taylor expansion around each point of  $\odot$  has a 1 for the first coefficient, a 0 for the coefficients 2 through  $k$ , and is arbitrary for the remaining coefficients. To make this more precise we will include a short discussion on jet bundles—in particular, jets of elements in  $\text{SDiff}(M)$ .

**4.1. Jet Bundles of the Special Diffeomorphism Group.** Consider the following equivalence relation:  $\varphi_1, \varphi_2 \in \text{SDiff}(M)$  are *equivalent to  $k$ th order at  $\odot$*  if they have the same Taylor expansion at  $\odot$ . We denote the set of equivalence classes for this relation by  $\mathcal{J}_\odot^k(\text{SDiff}(M))$ . Elements of  $\mathcal{J}_\odot^k(\text{SDiff}(M))$  are called  $k$ -jets of  $\text{SDiff}(M)$  sourced at  $\odot$ . For  $k = 0$  this simply means that  $\varphi_1(\odot) = \varphi_2(\odot)$  and for  $k = 1$  this means  $T_\odot \varphi_1 = T_\odot \varphi_2$ . Given a  $\varphi \in \text{SDiff}(M)$ , we denote its  $k$ -jet by  $j_\odot^k \varphi$ . Additionally, there is a natural projection  $\pi_l^k : \mathcal{J}_\odot^k(\text{SDiff}(M)) \rightarrow \mathcal{J}_\odot^l(\text{SDiff}(M))$  for  $l < k$ . In particular, the projection  $\pi_0^k$  sends  $k$ -jets to  $N$ -tuples of points in  $M$  at the base of these jets via the map  $j_\odot^k(\varphi) \mapsto \varphi(\odot) \in X^{(0)}$ .

In addition, “jet” is a functor and it commutes with the “tangent functor  $T$ ”. This means

$$T\mathcal{J}_\odot^k(\text{SDiff}(M)) = \mathcal{J}_\odot^k(T\text{SDiff}(M)).$$

Given a vector-field  $u \in \mathfrak{X}_{\text{div}}(M)$  and an  $x^{(0)} \in X$ , we may consider the  $k$ -jet  $j_{x^{(0)}}^k(u)$  by viewing  $u$  as a tangent vector to  $\text{SDiff}(M)$  at the identity. Given

any  $\varphi \in \text{SDiff}(M)$  such that  $\varphi(\odot) = x^{(0)}$  we see that  $j_{\odot}^k(u \circ \varphi)$  is an element of  $T\mathcal{J}_{\odot}^k(\text{SDiff}(M))$ . Moreover, we may consider the entire equivalence class,  $x = j_{\odot}^k\varphi$ , as a set of maps which can be composed with the set of maps comprising  $j_{x^{(0)}}^k(u)$  to arrive at  $j_{x^{(0)}}^k(u) \circ x \equiv j_{\odot}^k(u \circ \varphi)$ . The notation used on the left hand side of this equation will be useful when comparing ideas in this section to those mentioned in the previous sections.

Finally, we can also consider the set  $\mathcal{J}_{\odot}^k(G_{\odot})$  which consists of equivalence classes restricted to elements of  $G_{\odot}$ . It should be clear that  $\mathcal{J}_{\odot}^k(G_{\odot})$  is a (finite dimensional) Lie group with the group multiplication  $j_{\odot}^k(\psi_1) \cdot j_{\odot}^k(\psi_2) := j_{\odot}^k(\psi_1 \circ \psi_2)$ . The Lie group  $\mathcal{J}_{\odot}^k(G_{\odot})$  acts on  $\mathcal{J}_{\odot}^k(\text{SDiff}(M))$  through the right action

$$j_{\odot}^k(\varphi) \cdot j_{\odot}^k(\psi) = j_{\odot}^k(\varphi \circ \psi)$$

for each  $\varphi \in \text{SDiff}(M)$  and  $\psi \in G_{\odot}$ . We will use this action in the context of Noether's theorem in §5 to find conserved momenta for a class of particle methods.

**4.2. Quotients and  $k^{\text{th}}$  order Interpolation Methods.** In this section we describe the quotient  $\text{SDiff}(M)/G_{\odot}^{(k)}$ .

**Proposition 4.1.** *The quotient  $X^{(k)} := \text{SDiff}(M)/G_{\odot}^{(k)}$  is given by  $k$ -jets of  $\text{SDiff}(M)$  sourced at  $\odot$ , i.e.,*

$$X^{(k)} \equiv \mathcal{J}_{\odot}^k(\text{SDiff}(M)).$$

*Proof.* Let  $\varphi \in \text{SDiff}(M)$ . Then we see that  $j_{\odot}^k\varphi = j_{\odot}^k(\varphi \circ \psi)$  for all  $\psi \in G_{\odot}^k$ . So, for each equivalence class  $[\varphi]$  there exists a unique  $k$ -jet. Conversely, a  $k$ -jet  $x \in \mathcal{J}_{\odot}^k(\text{SDiff}(M))$  can be associated to the equivalence class of some  $\varphi \in \text{SDiff}(M)$ , which is only unique up to an element of  $G_{\odot}^k$ , by the very definition of  $\mathcal{J}_{\odot}^k(\text{SDiff}(M))$ .  $\square$

**Example 4.1.** Upon choosing a frame  $\{f_1, \dots, f_n\}$  with unit volume over the points  $\odot_1, \dots, \odot_N$ , we may identify  $X^{(1)}$  as the set of  $N$ -tuples of unit volume frames with non-overlapping base points. If  $M = \mathbb{R}^d$  we can view unit volume frames as elements of  $\text{SL}(d)$  and we can view  $X^{(1)}$  as the trivial fiber-bundle over  $X^{(0)}$  given by the set  $X^{(0)} \times \text{SL}(d)^N$ . Through this observation It should be clear that  $X^{(1)}$  consists of particles which carry eccentricity and orientation. For  $k > 1$  we may interpret  $X^{(k)}$  as a fiber bundle over  $X^{(0)}$  where the fibers consist of infinitesimal perturbations of orientation and shape. Even less formally,  $X^{(k)}$  can be visualized as  $N$ -tuples of non-overlapping particles with orientation, shape, and an ability to “jiggle” like rubber to  $k^{\text{th}}$  order.

Let us return to the general case and follow the procedure of Lagrange-Poincaré reduction used in the last section but for a  $k^{\text{th}}$  order interpolation method, formally defined below.

**Definition 4.1.** *A  $k^{\text{th}}$  order interpolation method is a map,  $\mathcal{I} : TX^{(k)} \rightarrow \mathfrak{X}_{\text{div}}(M)$ , such that  $j_{x^{(0)}}^k(\mathcal{I}(\dot{x})) \circ x = (x, \dot{x})$  where  $(x, \dot{x}) \in TX^{(k)}$ , and  $x^{(0)} = \pi_0^k(x)$ .*

In other words, a  $k^{\text{th}}$  order interpolation method takes infinitesimally varying  $k$ -jets (given by  $\dot{x}$ ) to vector fields which match these infinitesimally variations  $k$ -jets at the points of the particles (given by  $x^{(0)}$ ). Even more concretely,  $k^{\text{th}}$  order interpolation methods approximate vector-fields to  $k^{\text{th}}$  order, given  $k^{\text{th}}$  order constraints at a finite set of points.

By using a  $k^{\text{th}}$  order interpolation method, we can follow the same sequence we used before by defining the isomorphism  $\Psi_{\mathcal{I}} : T\text{SDiff}(M)/G_{\odot}^{(k)} \rightarrow TX^{(k)} \oplus \tilde{\mathfrak{g}}_{\odot}^{(k)}$  given by

$$\Psi_{\mathcal{I}}([v_{\varphi}]) = j_{\odot}^k(v_{\varphi}) \oplus (v_{\varphi} \circ \varphi^{-1} - \mathcal{I}(v_{\varphi})).$$

From here the symbolic manipulations are entirely the same as in the previous sections. The reduced Lagrangian

$$[L](x, \dot{x}, \xi_x) = \frac{1}{2} \int_M \|\mathcal{I}(\dot{x}) + \xi_x\|^2 d\text{vol}(m).$$

is now defined on the Whitney sum,  $TX^{(k)} \oplus \tilde{\mathfrak{g}}_{\odot}^{(k)}$ . The Lagrange-Poincaré equations for the reduced Lagrangian system are described on this Whitney sum as well. The horizontal equation,

$$\frac{D}{Dt} \left( \frac{\partial[L]}{\partial \dot{x}} \right) - \frac{\partial[L]}{\partial x} = i_{\dot{x}} \tilde{B}_{\mu}$$

on  $TX^{(k)}$ , describes the motion of particles with extra structure (corresponding to  $k$ -jets of diffeomorphisms), and the vertical equation,

$$\frac{D}{Dt} \left( \frac{\partial[L]}{\partial \xi_x} \right) = -\text{ad}_{\xi_x}^* \left( \frac{\partial[L]}{\partial \xi_x} \right)$$

on fibers of the associated bundle,  $\tilde{\mathfrak{g}}_{\odot}^k$ , is formulated in terms of vector fields which vanish to  $k^{\text{th}}$  order at the particle locations. Additionally, the vector field  $\mathcal{I}(\dot{x})$  approximates the spatial velocity  $u = \dot{\varphi} \circ \varphi^{-1}$  to  $k^{\text{th}}$  order around the particles.

*Remark 4.1.* There is an extra symmetry in the horizontal equations. In particular, the group  $\mathcal{J}_{\odot}^k(G_{\odot})$  is a remaining symmetry left behind by  $G_{\odot}^{(k)}$ . If  $M = \mathbb{R}^d$  and  $k = 1$  we can identify  $\mathcal{J}_{\odot}^1(G_{\odot})$  with  $\text{SL}(d)^N$  by choosing unit volume frames above  $\odot_1, \dots, \odot_N$ .

If the Lagrangian does not depend on the orientation and shape of the particles, we get a right  $\text{SL}(d)^N$  symmetry which has yet to be “quotiented away”. This extra symmetry is attached to the particles, and results in conserved momenta as a consequence of Noether’s theorem. We will discuss this more in §5.2 in the context of a particle method.

## 5. APPLICATIONS TO PARTICLE METHODS

We hope that a computational scientist or engineer interested in particle methods for fluids finds the Lagrange-Poincaré equations thought provoking; we have



deliberately presented our approach using the “interpolation” point of view to encourage such thoughts from practitioners. In this section we describe two methods by which one can construct particle methods. This is not to say that these are the only methods one can do! Study along these lines is wide open for further exploration and implementation.

**5.1. A Variational Method.** The horizontal equations look very similar to the Euler-Lagrange equations except for the fact that there is a curvature force on the right hand side and the particle momenta,  $\frac{\partial[L]}{\partial \dot{x}}$ , depend on the vertical components  $(x, \xi_x) \in \tilde{\mathfrak{g}}_{\odot}^k$ . It is tempting to find a way of ignoring  $\xi_x$ . One way to do this is to add a fictitious non-holonomic constraint which restricts the spatial velocity  $u = \dot{\varphi} \circ \varphi^{-1}$  to be in the range of  $\mathcal{I}$ . That is to say, we introduce the constraint distribution  $\mathcal{D}^{\mathcal{I}} \subset T\text{SDiff}(M)$  by defining its fibers; the fiber over  $\varphi \in \text{SDiff}(M)$  is

$$\mathcal{D}_{\varphi}^{\mathcal{I}} := \{\mathcal{I}(\dot{x}) \circ \varphi \mid \dot{x} \in T_x X^{(k)}, x = j_{\odot}^k \varphi\}.$$

Then we restrict  $L$  to  $\mathcal{D}^{\mathcal{I}}$ . By construction, the non-holonomic force which keeps  $\dot{\varphi}$  in  $\mathcal{D}^{\mathcal{I}}$  is such that the vertical component  $\xi_x = \dot{\varphi} \circ \varphi^{-1} - \mathcal{I}^{(k)}(\dot{x})$  vanishes, where  $\dot{x} = \dot{\varphi}(\odot)$ . Thus the dynamics on  $X^{(k)}$  will determine the dynamics completely.

By construction, the constraint distribution  $\mathcal{D}^{\mathcal{I}}$  is  $G_{\odot}^{(k)}$ -invariant. This means that for each  $v_{\varphi} \in \mathcal{D}^{\mathcal{I}}$ , it follows that  $v_{\varphi} \circ \psi \in \mathcal{D}^{\mathcal{I}}$  for all  $\psi \in G_{\odot}^{(k)}$ . We denote the set of these orbits by  $[\mathcal{D}^{\mathcal{I}}]$ . Due to the injectivity of  $\mathcal{I}$ , there exists a vector bundle isomorphism  $\Phi_{\mathcal{I}} : TX^{(k)} \rightarrow [\mathcal{D}^{\mathcal{I}}]$  which sends  $\dot{x}$  to  $\mathcal{I}(\dot{x}) \circ x$  where we view  $x = \tau_{X^{(k)}}(\dot{x})$  as an equivalence class of diffeomorphism, so that it makes sense to compose it with another map. Given a  $G_{\odot}^{(k)}$ -invariant Lagrangian, it should be clear that  $\mathcal{I}$  induces a Lagrangian on  $TX^{(k)}$  given by  $L_X = [L]|_{\mathcal{D}^{\mathcal{I}}} \circ \Phi_{\mathcal{I}}$  where  $[L]$  is the  $G_{\odot}^{(k)}$ -reduced Lagrangian. We may use this construction to define a Lagrangian system on  $X^{(k)}$ , as in the following theorem.

**Theorem 5.1.** *Let  $L : T\text{SDiff}(M) \rightarrow \mathbb{R}$  be a  $G_{\odot}^{(k)}$ -invariant Lagrangian and let  $\varphi_t$  be a curve in  $\text{SDiff}(M)$ . Let  $\mathcal{D}^{\mathcal{I}}$  be a velocity constraint distribution induced by a  $k$ th order interpolation method,  $\mathcal{I}$ . Finally, let  $\varphi_t$  be a curve in  $\text{SDiff}(M)$  and set  $x(t) = j_{\odot}^k \varphi_t$ . If  $[L]$  is the reduced Lagrangian and*

$$(10) \quad L_X := L|_{\mathcal{D}^{\mathcal{I}}} \circ \Phi_{\mathcal{I}} : TX^{(k)} \rightarrow \mathbb{R},$$

*then the following statements are equivalent:*

- (1)  $\varphi_t$  satisfies the constrained Euler-Lagrange equations with respect to the velocity constraint distribution  $\mathcal{D}^{\mathcal{I}}$ .
- (2) The unreduced action  $S = \int_0^t L(\varphi, \dot{\varphi}) dt$  is extremized with respect to fixed end-point variations of the curve  $\varphi_t$  in restricted to the distribution  $\mathcal{D}^{\mathcal{I}}$ .
- (3) The action  $S_X = \int_0^t L_X(x, \dot{x}) dt$  is extremized with respect to arbitrary variations of the curve  $x(t)$  with fixed endpoints.

- (4) *The curve  $x(t)$  satisfies the Euler-Lagrange equations on  $X^{(k)}$  with respect to  $L_X$ .*

*Proof.* Assume  $S$  is extremized with respect to variations in  $\mathcal{D}^\mathcal{I}$ . In other words,  $\delta \int L(\varphi, \dot{\varphi}) dt = 0$  with respect to variations  $\delta\varphi = \mathcal{I}(\delta x) \circ \varphi$  for some variation  $\delta x$  of the curve  $x(t)$ . This works, because each fiber  $\mathcal{D}_\varphi^\mathcal{I}$  is isomorphic to the tangent fiber  $T_x X^{(k)}$ . Since both  $L$  and  $\mathcal{D}^\mathcal{I}$  are  $G_\odot^{(k)}$ -invariant we find

$$\delta \int L(\varphi, \dot{\varphi}) dt = \delta \int [L](\varphi, \dot{\varphi}) dt,$$

where the variations on the right hand side are taken in  $[\mathcal{D}^\mathcal{I}]$  by using  $[\delta\varphi]$ . These variations are given explicitly by  $\Phi(\delta x)$ . Furthermore, the isomorphism  $\Phi_\mathcal{I} : TX^{(k)} \rightarrow [\mathcal{D}^\mathcal{I}]$ , allows us to verify the equations  $L(\varphi, \dot{\varphi}) = [L](\Phi_\mathcal{I}(\dot{x}))$ . However, the right hand side of this expression is  $L_X$ . Therefore, extremizing  $S_X$  with respect to arbitrary variations of  $x(t)$  is equivalent to extremizing the restriction of  $S$  to  $\mathcal{D}^\mathcal{I}$  with respect to variations in  $\mathcal{D}^\mathcal{I}$ . The equivalence with the constrained and unconstrained Euler-Lagrange equations is a well known result.  $\square$

Theorem 5.1 suggests that we can estimate the exact evolution of an ideal fluid by solving the Euler-Lagrange equations for the Lagrangian  $L_X = [L] \circ \Phi$  on the space  $X^{(k)}$  and then use the interpolation,  $\mathcal{I}$ , to approximate the spatial velocity field. This is a method for estimating the flow of an inviscid fluid using only particle positions and velocities, i.e., a *particle method*. The error of such a scheme may be measured by the magnitude of the Lagrangian parameter used in constraining the dynamics to  $\mathcal{D}^\mathcal{I}$ .

*Remark 5.2.* If  $M = \mathbb{R}^d$ , the constraint force to  $\mathcal{D}^\mathcal{I}$  is the (co)vector field  $\mathcal{I}(\dot{x}) \cdot \nabla(\mathcal{I}(\dot{x}))$ . Noticing that this is computed solely in terms of particle motion, suggests that we can take a norm of this quantity and use Grownwall's inequality to find an error bound for our particle method. Additionally, this quantity is generally a function solely of  $\dot{x}$  and, therefore, can be used as a stopping criterion in these integrators.

**5.2. Kelvin's Circulation Theorem for Particles.** Recall the remaining symmetry of the horizontal equation mentioned in Remark 4.1. If the  $k^{th}$  order interpolation method is compatible with this symmetry, then the numerical method suggested in Theorem 5.1 will also possess this symmetry. This is a valuable property for users who desire to preserve this qualitative property of the system. Additionally, it allows one to construct integrators that conserve momenta (via a discrete time Noether theorem); see [MW01] or [HLW02] for details. We formalize this statement in the following proposition.

**Proposition 5.1.** *Given the constraint distribution,  $\mathcal{D}^\mathcal{I}$ , induced by a  $k^{th}$  order interpolation method,  $\mathcal{I}$ , the quantity*

$$J_\xi(\dot{x}) = \langle \dot{x}, x \circ \xi \rangle_X$$

is conserved by the Euler Lagrange equations for the Lagrangian,  $L_X$ , defined by equation (10) for each  $\xi$  in the Lie-algebra of  $\mathcal{J}^k(G_\odot)$ .

*Proof.* As already mentioned,  $L_X$  is right invariant with respect to the  $\mathcal{J}^k(G_\odot)$ -action. The action of  $g \in \mathcal{J}^k(G_\odot)$  on an  $x \in X^k$  is given by composition of jets,  $x \mapsto x \circ g$ . Thus, the infinitesimal generator of  $\xi$  is given by the vector field  $x \mapsto x \circ \xi$ .

Therefore, Noether's theorem implies the desired result upon noting that the momentum,  $\frac{\partial L_X}{\partial \dot{x}}$ , is given by the one-form  $\langle \dot{x}, \cdot \rangle_X$ .  $\square$

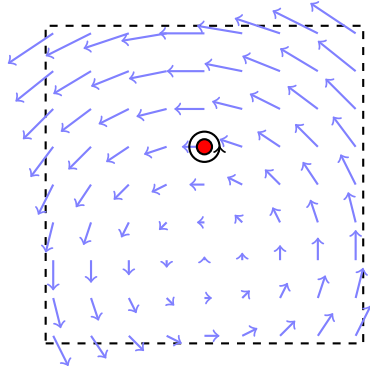


FIGURE 2. A particle-centric version of Kelvin's circulation theorem corresponds to conservation of circulation along infinitesimal curves circling the particles. For  $k = 1$  this implies that the spin and the rate of stretching are conserved.

Proposition 5.1 may be viewed as a particle-centric version of Kelvin's circulation theorem. In particular, it is known that the conserved quantity for the particle relabeling symmetry of an ideal fluid is given by the circulation. This implies that the integral of the dot product of the fluid velocity with an arbitrary vector field along an advected closed curve is conserved by the inviscid fluid equations [AK92, Chapter 1]. Proposition 5.1 tells us that curves which are infinitesimally close to the particles (see Figure 2) conserve the same quantity, even for the non-holonomically constrained version of the inviscid fluid equations described in Theorem 5.1. If  $k = 1$ , this implies that the flow will conserve the spin and the rate of deformation of each of the particles. For  $k > 1$  we can interpret the conserved quantities as manifestations of Kelvin's circulation for  $k^{th}$  order perturbations of these infinitesimal curves circling the particles. Therefore, when modeling the fluid using the *finite dimensional* Lagrangian system on  $X^{(k)}$  with Lagrangian  $L_X$ , one can better control the conservation of momenta and Kelvin's circulation theorem.

*Remark 5.3.* Variational particle methods for the EPDiff equation have been implemented and analyzed in [CDTM12] using the reconstruction mapping mentioned

in Remark 2.1. The convergence of this method was used to prove global existence and uniqueness in [CLP12]. Therefore, these ideas are not totally without precedent. However, the cited references are restricted to the case where  $k = 0$ .

**5.3. Spectral Methods and Hybrid Spectral-Particle Methods.** Building upon the ideas of the previous section, we can consider a method which is independent of the particle locations. To do this, choose  $N$  linearly independent vector fields,  $u_1, \dots, u_N \in \mathfrak{X}_{\text{div}}(M)$ . Then construct the interpolation method  $\mathcal{I} : TX^{(k)} \rightarrow \mathfrak{X}_{\text{div}}(M)$  given by

$$\mathcal{I}(\dot{x}) = \sum_i c_i(\dot{x}) u_i,$$

where the coefficients  $c_i$  are described implicitly by the constraints of definition 4.1 (it is a linear algebraic inverse problem). Using this interpolation method produces a spectral method since solving the Euler-Lagrange equations is equivalent to extremization over the constant vector space  $V = \text{span}(u_1, \dots, u_N)$  at each time step. This maybe counter intuitive, since spectral methods do not depend on the location of particles. One clear way to see the proposed method is *not* dependent on the locations of the particles is to note that given two different particle configurations  $x$  and  $y$  in  $X^{(0)}$ , the velocities  $\dot{x} = c_i u_i(x)$  and  $\dot{y} = c_i u_i(y)$  are both interpolated to the same velocity field, i.e.,  $\mathcal{I}(\dot{x}) = \mathcal{I}(\dot{y}) = c_i u_i$ . The method obtained by solving the Euler-Lagrange equations for the Lagrangian  $L_X$  defined in equation (10), is equivalent to choosing the coefficients  $c_i$  at each time in a manner guaranteeing that the action is extremized. In discrete time a variational integrator would generate the constants  $c_i$  at each time step in such a that they do not depend on the particle positions; but only on the previous group of  $c_i$ .

*Remark 5.4.* One should probably choose the basis  $\{u_1, \dots, u_N\}$  in such a way that as the number of particles goes to infinity, the basis captures a function space of decreasing regularity, e.g., using a Fourier basis on  $M = \mathbb{R}^d$ .

Judicious choices of bases lead to nice error bounds. One does not need to choose a full basis, however. One could choose vector fields  $u_1, \dots, u_k$  for  $k < N$  to construct an interpolation method which is partially spectral and partially dependent on particle locations. This can be considered a “multiscale” model in the sense that the spectral basis keeps track of the “large scale dynamics” while the remaining degrees of freedom are described explicitly by the motion of particles and can be thought to model the “fine scale dynamics”.

**5.4. Vortex Methods.** It was shown in [OS01] that Chorin’s vortex blob algorithm (originally developed for simulating inviscid fluids) provides an exact solution to the equations of motion for an ideal, inviscid, homogeneous, incompressible second grade fluid. More specifically, Chorin’s vortex blob algorithm yields solutions to the Lagrangian system on  $\text{SDiff}(M)$  with the reduced Lagrangian given

in terms of the spatial velocity by

$$[L]_\alpha(u) := \frac{1}{2} \int_{\mathbb{R}^2} (\|u\|^2 + \alpha \cdot \text{trace}(\nabla u^T \nabla u)) d^2x,$$

for some  $\alpha > 0$ . By construction, the vortex blob method conserves circulation by advecting gaussian blobs of vorticity. In this section we will describe in words how the vortex blob method may be viewed as a particular case of the algorithm described in this paper. Consider a  $1^{st}$  order interpolation method,  $\mathcal{I}$ , such that the spin of each particle is mapped by  $\mathcal{I}$  to the spatial velocity field of a vortex blob, and the translational component of the particles is mapped by  $\mathcal{I}$  to a spatial velocity field with 0 vorticity above each particle. Then, by Proposition 5.1, if we initialize the system with a spatial velocity field consisting solely of vortex blobs located above the particles, it follows that  $\mathcal{I}(x)$  remains a sum of vortex blobs for all time. Additionally, the particles will be advected by these vortex blobs and the vorticities will extremize the action of  $[L]_\alpha$ . As mentioned at the beginning of this section, this is precisely the evolution of the vortex blob algorithm.

Additionally, there are variants of the vortex blob method which allow for deformations of vortex blobs, as would be the case of an algorithm induced by a  $k^{th}$  order interpolation method for  $k > 1$ . For example, [UWB12] uses Hermite functions to generate a basis of functions above each vortex blob to do just this. It is not immediately clear whether this method satisfies a variational principle and therefore it is not known whether such a method matches the framework described in this paper. Nonetheless, appealing to higher-order deformations of vortices did yield higher accuracy in numerical experiments for the case  $k = 2$ . Another interesting method potentially linked to our approach, is the adaptive smoothed particle hydrodynamics proposed in [SMVO96].

## 6. EXTENSIONS TO OTHER FLUIDS

Given the structure presented so far, the process of extending these ideas to other types of ideal fluids can be summarized with little effort. In this section, we will provide a sketch of how to extend the constructions of the previous section to complex fluids and Euler- $\alpha$  models.

**6.1. Complex Fluids.** Consider the set  $\mathcal{F}(M, \mathcal{O})$  of smooth maps from a manifold  $M$  to a Lie group  $\mathcal{O}$ . The Lie group  $\text{SDiff}(M)$  acts on  $\mathcal{F}(M, \mathcal{O})$  on the right by the assignment  $f \in \mathcal{F}(M, \mathcal{O}) \mapsto f \circ \varphi \in \mathcal{F}(M, \mathcal{O})$  for each  $\varphi \in \text{SDiff}(M)$ . This right action defines the semi-direct product Lie group  $\text{SDiff}(M) \ltimes \mathcal{F}(M, \mathcal{O})$ , where the group multiplication is  $(\varphi_1, f_1) \cdot (\varphi_2, f_2) := (\varphi_1 \circ \varphi_2, (f_1 \circ \varphi_2) \cdot f_2)$ . A unifying framework for ideal complex fluids was described in [GBR09] by performing Euler-Poincaré reduction on a Lagrangian system with the configuration manifold  $\text{SDiff}(M) \ltimes \mathcal{F}(M, \mathcal{O})$ . The number of scenarios captured by this framework is truly remarkable (magnetohydrodynamics, spin-glass fluids, liquid crystals, fluids

with microstructure, Yang-Mills fluids, etc.). It should be clear that one may reduce such a system by  $G_{\odot}^{(k)}$  as well. This could be particularly fruitful for the case of fluids with microstructure, where the local orientation and shape of the fluid particles shows up in the Lagrangian. For example, in the case of micromorphic nematic liquid crystals,  $\mathcal{O} = \text{SO}(3)$ . We see that  $X^{(1)}$  for  $M = \mathbb{R}^3$  is the trivial bundle  $(\mathbb{R}^3)^N \times \text{SL}(3)^N$ . Since  $\text{SO}(3)$  is a subgroup of  $\text{SL}(3)$  it follows that a method on  $X^{(1)}$  could be capable of modeling liquid crystals.

**6.2. 2nd Grade Fluids and other generalizations.** A fairly modest generalization to the preceding setup is to add gradient terms to the Lagrangian and consider second grade fluids. For example, the kinetic energy on  $M = \mathbb{R}^3$  given by

$$L_{\alpha}(\varphi, \dot{\varphi}) = \frac{\rho}{2} \int (\|u\|^2 + \alpha \cdot \text{trace}(\nabla u^T \nabla u)) d^3x$$

where  $u = \dot{\varphi} \circ \varphi^{-1}$  and  $\alpha > 0$ , clearly exhibits particle relabeling symmetry and leads to the Euler- $\alpha$  model in the incompressible case. We may even consider the Lagrangian

$$L_{\mathbb{I}}(\varphi, \dot{\varphi}) = \frac{1}{2} \int u \cdot \mathbb{I}(u) d^3x,$$

where  $\mathbb{I}$  is a positive definite differential operator on  $\mathfrak{X}(M)$  (again, the Helmholtz operator  $I - \alpha \cdot \nabla$  is a good example). The momentum is given by  $G * u$  where  $G$  is the Green's function for  $\mathbb{I}$ . In this case, we may even consider compressible fluids if we use interpolation methods which map to the full set of smooth vector fields on  $M$  (see [HSS09, Chapter 11] for a good overview). If  $G$  is non-singular, the interpolation method mentioned in remark 2.1 becomes a particularly natural choice.

## 7. CONCLUSION

Since the publication of [Arn66], it has been well known that Euler's equations of motion for an ideal, inviscid, incompressible, homogeneous fluid may be regarded as Euler-Poincaré equations on the Lie algebra of divergence free vector fields tangent to the boundary. Therefore, it has been clear that one could reduce by various subgroups of the diffeomorphism group. In this paper we have accomplished one version of this reduction, using the notion of an interpolation method. We chose to use interpolation methods instead of principal connections (as in [CMR01]) to clarify the notion of estimating the velocity field of the fluid with *particles*. Specifically, if an interpolation method is chosen, one may estimate the spatial velocity field of the fluid and even estimate the evolution of the system over short times by integrating a non-holonomically constrained version of the equations. We also discussed the reduction process for higher-order isotropy groups. The horizontal equations for higher-order isotropy reduction contain extra symmetry. Using a particle method which respects this symmetry produced a discrete version of Kelvin's circulation theorem.

Future work includes:

- Implementing an instance of the particle method described in this paper for  $k \geq 1$  and comparing performance with the vortex blob method.
- Extending these constructions to other types of fluid, particularly complex fluids.
- Appending a finite dimensional model of the vertical Lagrange-Poincaré equations would yield a finite dimensional model of a fluid on a Lie groupoid. This could generalize the integrator described in [GMP<sup>+</sup>11] which models fluid on a finite dimensional Lie group.

## APPENDIX A. INTERPOLATION METHODS AND PRINCIPAL CONNECTIONS

In this appendix our goal is to prove that

$$\tilde{B}(\dot{x}, \delta x) = \mathcal{I}([\mathcal{I}(\dot{x}), \mathcal{I}(\delta x)](x)) - [\mathcal{I}(\dot{x}), \mathcal{I}(\delta x)]$$

where  $\tilde{B}$  is the reduced curvature tensor defined in equation (6). In the process, we will also clarify the “equivalence” of interpolation methods with principal connections on the right principal bundle  $\pi : \text{SDiff}(M) \rightarrow X$ . First, we use the definition of a (right) principal connection (as in [MMR90], but adapted for the case of right invariance).

**Definition A.1.** *Let  $\pi : Q \rightarrow S$  be a right principal bundle with structure group  $G$ . Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . A right equivariant principal connection on  $Q$  is a  $\mathfrak{g}$ -valued one-form  $A \in \Omega^1(Q, \mathfrak{g})$ , satisfying the following properties:*

- (1) *For each  $\xi \in \mathfrak{g}$ , we have  $A(\xi_Q) = \xi$ , where  $\xi_Q \in \mathfrak{X}(Q)$ , defined by  $\xi_Q(q) := \left. \frac{d}{dt} \right|_{t=0} q \cdot \exp(t\xi)$  for any  $q \in Q$ , is the infinitesimal generator of  $\xi$ .*
- (2) *For each  $g \in G$  and  $v \in TQ$  we have  $A(g \cdot v) = \text{Ad}_g^{-1}(A(v))$ .*

In the context of the principal bundle  $\text{SDiff}(M)$  over  $X$ , the notion of a right equivariant principal connection is identical to the notion of an interpolation method. This is shown in the following lemma.

**Lemma A.1.** *Given an interpolation method  $\mathcal{I} : TX \hookrightarrow \mathfrak{X}_{\text{div}}(M)$  the map*

$$(11) \quad A(v_\varphi) = T\varphi^{-1} \circ v_\varphi - \varphi^* \mathcal{I}(v_\varphi(\odot))$$

*is a right equivariant principal connection on the  $G_\odot$ -principal bundle  $\pi_X : \text{SDiff}(M) \rightarrow X$ .*

*Proof.* We must prove three things.

- (1) Show that  $A$  maps to the Lie algebra of  $G_\odot$ .
- (2) Show that  $A$  satisfies the first property of a principal connection.
- (3) Show that  $A$  satisfies the second property (i.e., right equivariance).

To show that  $A$  maps to the Lie algebra of  $G_\odot$  is equivalent to proving that the values of  $A$  are divergence free vector fields vanishing at all points of  $\odot \subset M$ . Clearly,  $T\varphi^{-1} \circ v_\varphi$  is a divergence free vector-field on  $M$ , as it is merely the left trivialization of  $v_\varphi$ , and  $\varphi^*\mathcal{I}(v_\varphi(\odot))$  is a divergence free vector field by construction. Thus, the difference of the two, namely  $A(v_\varphi) = T\varphi^{-1} \circ v_\varphi - \varphi^*(\mathcal{I}(v_\varphi(\odot)))$ , is a divergence free vector field as well. Additionally, we have  $A(v_\varphi)(\odot) = T\varphi^{-1}(v_\varphi(\odot)) - T\varphi^{-1}(v_\varphi(\odot)) = 0$ , where we have used the property  $\mathcal{I}(v_x)(x) = v_x$  for all  $v_x \in T_x X$ . This proves (1).

To prove (2), let  $\xi \in \mathfrak{g}_\odot$ . The infinitesimal generator of  $\xi$  is the map  $\varphi \mapsto T\varphi \circ \xi$ .

It is now easy to check that  $A(T\varphi \circ \xi) = \xi$  for all  $\xi \in \mathfrak{g}_\odot$ .

Finally, to prove (3), note that for any  $\psi \in G_\odot$  and  $v_\varphi \in T\text{SDiff}(M)$  we have

$$\begin{aligned} A(v_\varphi \circ \psi) &= T(\varphi \circ \psi)^{-1} \circ (v_\varphi \circ \psi) - (\varphi \circ \psi)^*\mathcal{I}((v_\varphi \circ \psi)(\odot)) \\ &= \psi^*(T\varphi^{-1} \circ v_\varphi - \varphi^*\mathcal{I}(v_\varphi(\odot))) \\ &= \text{Ad}_\psi^{-1}(A(v_\varphi)) \end{aligned}$$

since the adjoint action is given by push-forward.  $\square$

Conversely, an arbitrary right equivariant principal connection,  $A \in \Omega^1(\text{SDiff}(M), \mathfrak{g}_\odot)$ , naturally induces an interpolation method. This is done as follows. Recall that the kernel of  $A$  is a horizontal subbundle of  $T\text{SDiff}(M)$  which naturally defines the *horizontal lift* operator  $h^\uparrow : TX \oplus \text{SDiff}(M) \rightarrow T\text{SDiff}(M)$ . An interpolation method,  $\mathcal{I}$ , is obtained by composing this horizontal lift with the right trivialization of  $T\text{SDiff}(M)$ , i.e.,

$$h_\varphi^\uparrow(\dot{x}) \equiv \mathcal{I}(\dot{x}) \circ \varphi.$$

The above equation defines  $\mathcal{I}$  uniquely. Moreover, the principal connection obtained from  $\mathcal{I}$  in equation (11) is precisely  $A$ . Thus the concepts of principal connections and an interpolation methods are effectively equivalent for the principal bundle  $\pi : \text{SDiff}(M) \rightarrow X$ .

It is shown in [MMR90] that the reduced curvature tensor (6) may also be given by the expression

$$\tilde{B}(\dot{x}, \delta x) = \varphi_* A([h_\varphi^\uparrow(\dot{x}), h_\varphi^\uparrow(\delta x)]),$$

where the choice of  $\varphi$  is arbitrary up to the constraint  $\varphi(\odot) = x = \tau_X(\dot{x})$ . This version of the reduced curvature tensor allows us to prove Proposition 3.3 in view of the following lemma.

**Lemma A.2.** *Let  $[\cdot, \cdot]_M$  be the Lie bracket of vector fields on  $M$  and  $[\cdot, \cdot]_{\text{SDiff}(M)}$  the Lie bracket of vector fields on  $\text{SDiff}(M)$ . Let  $\rho_{\text{triv}} : T\text{SDiff}(M) \rightarrow \mathfrak{X}_{\text{div}}(M)$  be the right trivializing morphism  $\delta\varphi \mapsto \delta\varphi \circ \varphi^{-1}$ . Then, for  $X, Y \in \mathfrak{X}(\text{SDiff}(M))$ , we have*

$$[\rho_{\text{triv}} \circ X, \rho_{\text{triv}} \circ Y]_M = \rho_{\text{triv}} \circ [X, Y]_{\text{SDiff}(M)}.$$



*Proof.* Using the dynamic definition of Lie derivative and evaluating at a given diffeomorphism  $\varphi \in \text{SDiff}(M)$  we have

$$[X, Y]_{\text{SDiff}(M)}(\varphi) = \partial_t \partial_s \varphi_{t,s} - \partial_s \partial_t \varphi_{t,s}$$

where  $\partial_s = \frac{d}{ds}|_{s=0}$ ,  $\partial_t = \frac{d}{dt}|_{t=0}$  and  $\varphi_{s,t}$  is such that  $\varphi_{0,0} = \varphi$ ,  $\partial_t \varphi_{t,0} = X(\varphi)$  and  $\partial_s \varphi_{0,s} = Y(\varphi)$ . Applying  $\rho_{\text{triv}}$ , we find

$$\begin{aligned} \rho_{\text{triv}}([X, Y]_{\text{SDiff}(M)}(\varphi)) &= \partial_t \partial_s (\varphi_{t,s} \circ \varphi^{-1}) - \partial_s \partial_t (\varphi_{t,s} \circ \varphi^{-1}) \\ &= [\rho_{\text{triv}}(X(\varphi)), \rho_{\text{triv}}(Y(\varphi))]_M \end{aligned}$$

thus proving the statement.  $\square$

The take-away message from this lemma is that for each  $\varphi \in \text{SDiff}(M)$

$$[X, Y]_{\text{SDiff}(M)}(\varphi) = [X(\varphi) \circ \varphi^{-1}, Y(\varphi) \circ \varphi^{-1}]_M \circ \varphi$$

for  $X, Y \in \mathfrak{X}(\text{SDiff}(M))$ . The right-hand side only depends on the values of  $X$  and  $Y$  at  $\varphi \in \text{SDiff}(M)$  and provides a bracket on the vector space  $T_\varphi \text{SDiff}(M)$ , as opposed to the set of vector fields  $\mathfrak{X}(\text{SDiff}(M))$ . With Lemma A.2, we are now prepared to prove Proposition 3.3.

*Proof.* Using Lemma A.2 we may free ourselves of the burden of extending vectors in  $\text{SDiff}(M)$  to vector fields. This allows us to make the calculation of the reduced curvature tensor as follows.

$$\begin{aligned} \tilde{B}(\dot{x}, \delta x) &= \varphi_* A([h_\varphi^\uparrow(\dot{x}), h_\varphi^\uparrow(\delta x)]) \\ &= \varphi_*(T\varphi^{-1} \circ [\mathcal{I}(\dot{x}) \circ \varphi, \mathcal{I}(\delta x) \circ \varphi] - \mathcal{I}([\mathcal{I}(\dot{x}), \mathcal{I}(\delta x)](x))) \\ &= [\mathcal{I}(\dot{x}), \mathcal{I}(\delta x)] - \mathcal{I}([\mathcal{I}(\dot{x}), \mathcal{I}(\delta x)](x)) \end{aligned}$$

which proves the proposition.  $\square$

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